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Soliton solutions for quasilinear Schrödinger equations, II

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Abstract

For a class of quasilinear Schrödinger equations, we establish the existence of ground states of soliton-type solutions by a variational method.

Keywords: Standing waves; Quasilinear Schrödinger equations; Orlicz spaces; Minimax methods

1. Introduction

We study the ground states solutions for the following type quasilinear elliptic equations in the entire space

$$-\Delta u + V(x)u - (\Delta(|u|^2))u = f(u) \quad \text{in } \mathbf{R}^N,$$

i.e., we are seeking positive solutions with least energy. Solutions of this type are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t z = -\Delta z + V(x)z - f(|z|^2)z - \kappa\Delta h(|z|^2)h'(|z|^2)z, \tag{1}$$

where $V = V(x)$, $x \in \mathbf{R}^N$, is a given potential, κ is a real constant and f, h are real functions of essentially pure power forms. Semilinear case corresponding to $\kappa = 0$ has been studied extensively in recent years (e.g., [6,14,31]). Quasilinear equations of

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form (1) appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of h. The case of $h(s) = s$ was used for the superfluid film equation in plasma physics by Kurihara [18] (cf. [19]). In the case $h(s) = (1+s)^{1/2}$, Eq. (1) models the self-channeling of a high-power ultrashort laser in matter, see [7,9,12,30] and the references in [8]. Eq. (1) also appears in plasma physics and fluid mechanics [18,19,21,24,26], in the theory of Heisenberg ferromagnets and magnons [5,17,20,27,32], in dissipative quantum mechanics [15] and in condensed matter theory [23]. In the mathematical literature, very few results are known about equations of the form (1).

In this paper, we consider standing wave solutions for quasilinear Schrödinger equations of form (1) with h and f being pure power functions of the dependent variable. But we want to mention that our method applies to more general type of nonlinearity and we refer to remarks at the end of the paper. As a model, let us consider the case $h(s) = s$, $f(s) = \lambda s^{\frac{p-1}{2}}$ and $\kappa > 0$. Putting $z(t) = \exp(-iFt)u(x)$ we obtain a corresponding equation of elliptic type which has a formal variational structure:

$$-\Delta u + V(x)u - \frac{1}{2}(\Delta(|u|^2))u = \lambda |u|^{p-1}u, \quad u > 0 \text{ in } \mathbf{R}^N. \tag{2}$$

Here, we have renamed $V(x) - F$ to be $V(x)$ and without loss of generality, we assume $\kappa = \frac{1}{2}$.

In the following, we always assume $V \in C(\mathbf{R}^N, \mathbf{R})$ and $\inf_{\mathbf{R}^N} V(x) > 0$.

In [22,25], a constrained problem associated with Eq. (2) has been considered. Namely, define

$$m = \inf_M E(u), \tag{3}$$

where

$$M = \{u \in X \mid ||u||_{p+1} = 1\}$$

$$E(u) = \int_{\mathbf{R}^N} (|\nabla u|^2 + Vu^2)\, dx + \int_{\mathbf{R}^N} |u|^2 |\nabla u|^2\, dx$$

and

$$X = \left\{ u \in H^1(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} V(x)u^2\, dx < \infty \right\}$$

is a closed subspace of $H^1(\mathbf{R}^N)$ depending upon the potential $V(x)$. Under suitable conditions, it was proved that Eq. (2) has a positive solution for a sequence of $\lambda_n \to \infty$ and a sequence of $\lambda_n \to 0$. This was done mostly for $N = 1$ in [25] and for higher dimensional cases in [22]. However, the question whether there exists solutions for any prescribed λ was left open in these paper. We shall address this question in this paper and give existence of positive solutions for Eq. (2) for any prescribed $\lambda > 0$.

We consider several type of potentials.

(V1) $\lim_{|x|\to\infty} V(x) = +\infty$.

(V2) V is radially symmetric, i.e., $V(x) = V(|x|)$.

(V3) V is periodic in each variable of $x_1, \ldots, x_N$.

(V4) $V_\infty := \lim_{|x|\to\infty} V(x) = ||V||_{L^\infty(\mathbf{R}^N)} < \infty$.

Let $2^* = \frac{2N}{N-2}$ for $N \geqslant 3$, $2^* = \infty$ for $N = 1, 2$.

Theorem 1.1. *Let* $4 \leqslant p+1 < 22^*$. *Then for any* $\lambda > 0$, (2) *has a positive solution, provided that one of the following four conditions hold*: (V1); (V2) *and* $N \geqslant 2$; (V3); (V4).

To deal with this type of problems, difficulties lie in two aspects. On one hand, there are three different scales in the equation, which causes problems in using the constrained method. In general, one cannot scale the minimizer of the constrained problem into a solution of Eq. (2). On the other hand, for the unconstrained problem there is no natural functions spaces for the associated energy functional to be well defined and this is due to the super-critical growth condition ($p+1$ can be greater than 2^*) on the nonlinearity. One might treat the unconstrained problem by approximations with subcritical nonlinearities, giving solutions for the subcritical problems. However, this will not produce solutions for $2^* \leqslant p+1 < 22^*$ which seems to be the natural growth condition for the nonlinearity as shown by the minimization argument in [22].

In Section 2, we introduce a new formulation of the problem by using a change of variables and we shall reformulate the problem. The novelty of our paper is that we treat the new problem in an Orlicz space and this enables us to handle the nonlinearity in a uniform way. Finally, Theorem 1.1 will be proved in Section 3.

2. Reformulation of the problem and preliminaries

For simplicity, we assume $\inf_{\mathbf{R}^N} V(x) = 1$. We may formally formulate our problem in a variational setting as follows: consider

$$J(u) = \frac{1}{2}\int_{\mathbf{R}^N} (1+u^2)|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbf{R}^N} V(x)u^2\,dx - \frac{1}{p+1}\int_{\mathbf{R}^N} |u|^{p+1}\,dx$$

for $u \in H^1(\mathbf{R}^N)$ or the closed subspace of $H^1(\mathbf{R}^N)$

$$X = \left\{ u \in H^1(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} V(x)u^2\,dx < \infty \right\}.$$

In case (V2), it is understood that X contains only the radially symmetric functions. But the difficulty is the differentiability of J, in fact under our growth condition of the nonlinearity J is not even defined in X.

The new idea of this paper is to make a change of variables first:

$$dv = \sqrt{1+u^2}\,du, \quad v = h(u) = \tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u+\sqrt{1+u^2}\right).$$

Then h is strictly monotone and has an inverse function: $u = f(v)$. We observe that

$$h(u) \sim \begin{cases} u, & |u| \ll 1, \\ \frac{1}{2}u|u|, & |u| \gg 1, \end{cases} h'(u) = \sqrt{1+u^2}, \tag{4}$$

$$f(v) \sim \begin{cases} v, & |v| \ll 1, \\ \sqrt{\frac{2}{|v|}}v, & |v| \gg 1, \end{cases} f'(v) = \frac{1}{h'(u)} = \frac{1}{\sqrt{1+u^2}} = \frac{1}{\sqrt{1+f^2(v)}}. \tag{5}$$

Also for some $C_0 > 0$ it holds

$$G(v) = f^2(v) \sim \begin{cases} v^2, & |v| \ll 1, \\ 2|v|, & |v| \gg 1, \end{cases} G(2v) \leqslant C_0 G(v), \tag{6}$$

$$G(v) \text{ is convex}, \quad G'(v) = \frac{2f(v)}{\sqrt{1+f^2(v)}}, \quad G''(v) = \frac{2}{(1+f^2(v))^2} > 0. \tag{7}$$

Using this change of variable, we can rewrite the functional $J(u)$ as

$$\begin{aligned} I(v) = &\frac{1}{2}\int_{\mathbf{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbf{R}^N} V(x) f^2(v)\,dx \\ &- \frac{1}{p+1}\int_{\mathbf{R}^N} |f(v)|^{p+1}\,dx. \end{aligned} \tag{8}$$

I is defined on the space

$$H_G^1 = \left\{ v \mid \int_{\mathbf{R}^N} |\nabla v|^2\,dx < \infty,\ \int_{\mathbf{R}^N} V(x)G(v)\,dx < \infty \right\}.$$

Here again, in case (V2) only radially symmetric functions are in this space. We introduce the Orlicz space (e.g., [29])

$$E_G = \left\{ v \mid \int_{\mathbf{R}^N} V(x)G(v)\,dx < \infty \right\}$$

equipped with the following norm:

$$|v|_G = \inf_{\xi>0} \xi\left(1 + \int_{\mathbf{R}^N} V(x)G(\xi^{-1}v(x))\,dx\right),$$

and define the norm of H^1_G by

$$||v|| = |\nabla v|_{L^2(\mathbf{R}^N)} + |v|_G.$$

We collect some related facts. In the following, we use C to denote any constant that is independent of the sequences considered.

Proposition 2.1. (1) *E_G is a Banach space.*

(2) *Let $v_n \to v$ in E_G, then $\int_{\mathbf{R}^N} V(x)|G(v_n) - G(v)|\,dx \to 0$, $\int_{\mathbf{R}^N} V(x)|f(v_n) - f(v)|^2\,dx \to 0$.*

(3) *If $v_n \to v$ a.e. and $\int_{\mathbf{R}^N} V(x)G(v_n)\,dx \to \int_{\mathbf{R}^N} V(x)G(v)\,dx$, then $v_n \to v$ in E_G.*

(4) *The dual space $E_G^* = L^\infty \cap L_V^2 = \{w \mid w \in L^\infty, \int_{\mathbf{R}^N} V(x)w^2\,dx < \infty\}$.*

(5) *If $v \in E_G$, then $w = G'(v) = 2f(v)f'(v) \in E_G^*$, and $|w|_{E_G^*} = \sup_{|\phi|_G \leqslant 1} \langle w, \phi \rangle \leqslant C_1(1 + \int_{\mathbf{R}^N} V(x)G(v)\,dx)$, where C_1 is a constant independent of v.*

Proof. Since G satisfies the Δ_2-condition [29], $(E_G, |\cdot|_G)$ is a separable Banach space. The proof of (2) and (4) are elementary and we omit them here.

The proof of (3). It suffices to prove $\int_{\mathbf{R}^N} V(x)G(v_n - v)\,dx \to 0$. In fact, by the Δ_2-condition, $\forall \xi \in (0,1)$, there exists N such that for $n \geqslant N$, $\int_{\mathbf{R}^N} V(x)G(\xi^{-1}(v_n - v))\,dx \leqslant 1$. Hence, $\xi(1 + \int_{\mathbf{R}^N} V(x)G(\xi^{-1}(v_n - v))\,dx) \leqslant 2\xi$. By the definition of $|\cdot|_G$, we have $|v_n - v|_G \leqslant 2\xi, \forall n \geqslant N$. Now we are to prove

$$\int_{\mathbf{R}^N} V(x)G(v_n - v)\,dx \to 0 \text{ as } n \to \infty.$$

By Fatou's lemma, $\forall A \subset \mathbf{R}^N$, we have

$$\int_A V(x)G(v)\,dx \leqslant \liminf_{n\to\infty} \int_A V(x)G(v_n)\,dx$$

and

$$\int_{\mathbf{R}^N\setminus A} V(x)G(v)\,dx \leqslant \liminf_{n\to\infty} \int_{\mathbf{R}^N\setminus A} V(x)G(v_n)\,dx.$$

But $\int_{\mathbf{R}^N} G(v)\,dx = \lim_{n\to\infty} \int_{\mathbf{R}^N} V(x)G(v_n)\,dx$, hence

$$\int_A V(x)G(v)\,dx = \lim_{n\to\infty} V(x)G(v_n)\,dx \quad \forall A \subset \mathbf{R}^N.$$

We claim $V(x)G(v_n)$ is asymptotically equi-continuous in integration, that is, $\forall \varepsilon > 0$, there exist δ, N such that if $A \subset \mathbf{R}^N$, $|A| \leqslant \delta$ and $n \geqslant N$, then $\int_A V(x)G(v_n) \leqslant \varepsilon$. By the equi-continuity of the integral, there is a constant $\delta > 0$ such that if $A \subset \mathbf{R}^N$ with

$|A| \leqslant \delta$ then $\int_A V(x)G(v)\,dx \leqslant \frac{1}{2}\varepsilon$. Suppose the claim is false for ε. Then for $\delta_j = \frac{\delta}{2^j}, j = 1, 2, \ldots,$ there exist $A_j \subset \mathbf{R}^N$ and $n_j \geqslant j$ such that $|A_j| \leqslant \delta_j$ and $\int_{A_j} V(x)G(v_j)\,dx \geqslant \varepsilon$. Set $A = \bigcup_{j=1}^{\infty} A_j$, then $|A| \leqslant \sum_{j=1} |A_j| \leqslant \delta$. Hence, $\int_A V(x)G(v)\,dx \leqslant \frac{\varepsilon}{2}$. But $\int_A V(x)G(v_{n_j})\,dx \geqslant \int_{A_j} V(x)G(v_{n_j})\,dx \geqslant \varepsilon$, $\int_A V(x)G(v)\,dx = \lim_{j\to\infty} \int_A V(x)G(v_{n_j})\,dx \geqslant \varepsilon$, a contradiction. Now we take a bounded set A with $\int_{\mathbf{R}^N\setminus A} V(x)G(v)\,dx < \varepsilon$. We have $\int_{\mathbf{R}^N\setminus A} V(x)G(v_n)\,dx < \varepsilon$ for all n large. Hence

$$\begin{aligned}
&\int_{\mathbf{R}^N\setminus A} V(x)G(v_n - v)\,dx \\
&\quad\leqslant \frac{1}{2}\int_{\mathbf{R}^N\setminus A} V(x)(G(2v_n) + G(2v))\,dx \\
&\quad\leqslant C_0 \int_{\mathbf{R}^N\setminus A} V(x)(G(v_n) + G(v))\,dx \\
&\quad\leqslant C_0\varepsilon.
\end{aligned} \tag{9}$$

For the integral $\int_A V(x)\,G(v_n - v)\,dx$ we divide A into two subsets:

$$A_{n,1} = \{x \in A \mid |v_n - v| \leqslant a\}, \quad A_{n,2} = \{x \in A \mid |v_n - v| > a\}.$$

By the dominated convergence theorem, $\int_{A_{n,1}} V(x)G(v_n - v)\,dx \to 0$. The subset $A_{n,2}$ is small for a large since $|A_{n,2}|G(a) \leqslant C\int V(x)G(v_n - v)\,dx \leqslant C$. Hence, by our previous claim

$$\int_{A_{n,2}} V(x)G(v_n - v)\,dx \leqslant C_0 \int_{A_{n,2}} V(x)(G(v_n) + G(v))dx \leqslant C_0\varepsilon.$$

Altogether we get $\int_{\mathbf{R}^N} V(x)G(v_n - v)\,dx \to 0$.

The proof of (5). For $v \in E_G$, denote $g(v) = G'(v) = 2f(v)f'(v) = \frac{2f(v)}{\sqrt{1+f^2(v)}}$. We have $|g(v)| \leqslant 2$ and $g^2(v) \leqslant 4f^2(v)$. For all $\forall \phi \in E_G$, we have for all $\lambda > 0$

$$\begin{aligned}
\int_{\mathbf{R}^N} V(x)g(v)\phi\,dx &= \int_{\lambda|\phi|\leqslant 1} V(x)g(v)\phi\,dx + \int_{\lambda|\phi|\geqslant 1} V(x)g(v)\phi\,dx \\
&\leqslant \left(\int_{\mathbf{R}^N} V(x)g^2(v)\,dx\right)^{\frac{1}{2}} \left(\int_{\lambda|\phi|\leqslant 1} V(x)\phi^2\,dx\right)^{\frac{1}{2}} + 2\int_{\lambda|\phi|\geqslant 1} V(x)|\phi|\,dx \\
&\leqslant C\left(\int_{\mathbf{R}^N} V(x)f^2(v)\,dx + 1\right)\left(\frac{1}{\lambda}\left(\int_{\lambda|\phi|\leqslant 1} V(x)G(\lambda\phi)\,dx\right)^{\frac{1}{2}}\right.
\end{aligned}$$

$$+\frac{1}{\lambda}\int_{\lambda|\phi|\geqslant 1} V(x)G(\lambda\phi)\,dx\Bigg)$$

$$\leqslant \frac{C}{\lambda}\left(1+\int_{\mathbf{R}^N} V(x)G(v)\right)\left(1+\int_{\mathbf{R}^N} V(x)G(\lambda\phi)\right). \tag{10}$$

Hence

$$\int_{\mathbf{R}^N} V(x)g(v)\phi\,dx \leqslant C\left(1+\int_{\mathbf{R}^N} V(x)G(v)\,dx\right)|\phi|_G. \qquad \square$$

Proposition 2.2. (1) *The map*: $v \to f(v)$ *from* H_G^1 *into* $L^q(\mathbf{R}^N)$ *is continuous for* $2 \leqslant q \leqslant 22^*$.

(2) *Under* (V1), *the above map is compact for* $2 \leqslant q < 22^*$; *under* (V2) *with* $N \geqslant 2$, *this map is compact for* $2 < q < 22^*$; *under* (V3) *or* (V4), *the above map is compact for* $2 \leqslant q < 22^*$ *from* H_G^1 *into* $L^q_{\mathrm{loc}}(\mathbf{R}^N)$.

Proof. Let $u = f(v)$. Then it is easy to check that $||u||_X \leqslant ||v||$. By Sobolev embedding for $q \in [2, 2^*]$, $||f(v)||_{L^q(\mathbf{R}^N)} = ||u||_{L^q(\mathbf{R}^N)} \leqslant C||u||_X$ for some C depending on q and N. A direct computation shows that $\int_{\mathbf{R}^N} |\nabla(u^2)|^2\,dx = 4\int_{\mathbf{R}^N} \frac{f(v)^2|\nabla v|^2}{1+f(v)^2}\,dx \leqslant 4||v||^2$. By Sobolev inequality, $||f(v)||_{L^{22^*}} = ||u^2||_{L^{2^*}} \leqslant C||\nabla(u^2)||_{L^2} \leqslant C||v||^2$. Then by Hölder inequality, we get the map is continuous from H_G^1 into $L^q(\mathbf{R}^N)$ for $2 \leqslant q \leqslant 22^*$. Thus (1) is proved.

In case (V1), by the compact embedding from X into $L^q(\mathbf{R}^N)$ for $2 \leqslant q < 2^*$ we have $v \to f(v)$ is compact from H_G^1 into $L^q(\mathbf{R}^N)$ for $2 \leqslant q < 2^*$. Due to assertion (1) of this proposition and Hölder inequality again, we get that the map is compact from H_G^1 into $L^q(\mathbf{R}^N)$ for $2 \leqslant q < 22^*$. Similarly, one can prove the compactness (or local compactness) for other cases. We omit the details. $\square$

Proposition 2.3. (1) *I is well defined on* H_G^1.

(2) *I is continuous in* H_G^1.

(3) *I is Gauteaux-differentiable. For* $v \in H_G^1$, *the G-derivative* $I'(v)$ *is a continuous linear functional, and* $I'(v)$ *is continuous in v in the strong–weak topology, that is, if* $v_n \to v$ *strongly in* H_G^1, *then* $I'(v_n) \rightharpoonup I'(v)$ *weakly.*

Proof. To prove (1), we note that by Proposition 2.2(1), the embedding $v \to f(v)$, $H_G^1 \to L^{p+1}$ is continuous for $p+1 \leqslant \frac{4N}{N-2}$. Denote $w = |f|(v)f(v)$, then

$$\int_{\mathbf{R}^N} |\nabla w|^2\,dx = \int_{\mathbf{R}^N} \frac{4f^2(v)}{1+f^2(v)}|\nabla v|^2\,dx \leqslant C\int_{\mathbf{R}^N} |\nabla v|^2\,dx$$

and

$$\int_{\mathbf{R}^N} V(x)|w|\,dx = \int_{\mathbf{R}^N} V(x)G(v).$$

Then (1) follows from Sobolev embedding theorem and Proposition 2.1(2).

The proof of (2). Note that I consists of three terms. By Proposition 2.1, we need to check the superlinear term only.

$$\begin{aligned}
&\frac{1}{p+1}\int_{\mathbf{R}^N} |f|^{p+1}(v_n)\,dx - \frac{1}{p+1}\int_{\mathbf{R}^N} |f|^{p+1}(v)\,dx \\
&\quad = \int_0^1 dt \int_{\mathbf{R}^N} |f|^p(v+t(v_n-v))f'(v+t(v_n-v))(v_n-v)\,dx \\
&\quad \leqslant C\int_0^1 \int \mathbf{R}^N |f|^{p-1}(v+t(v_n-v))|v_n-v|\,dx \\
&\quad \leqslant C\int_0^1 dt\left(\int_{\mathbf{R}^N} |f|^{(p-1)\frac{2N}{N+2}}(v+t(v_n-v))\,dx\right)^{\frac{N+2}{2N}} \\
&\qquad \left(\int_{\mathbf{R}^N} |v_n-v|^{2^*}\,dx\right)^{\frac{1}{2^*}} \\
&\quad \leqslant C\left(\int_{\mathbf{R}^N} |f|^{(p-1)\frac{2N}{N+2}}(v)\,dx + \int_{\mathbf{R}^N} |f|^{(p-1)\frac{2N}{N+2}}(v_n)\,dx\right)^{\frac{N+2}{2N}} \\
&\qquad \times \|v_n-v\|_{L^{2^*}} \leqslant C\|v_n-v\|,
\end{aligned} \tag{11}$$

where $(p-1)\frac{2N}{N+2} < \frac{4N}{N-2}$ for $p+1 < \frac{4N}{N-2}$.

For (3) we consider the second and the third terms of the functional I. We see for $\phi \in E_G$

$$\begin{aligned}
&\frac{1}{t}\int_{\mathbf{R}^N} V(x)(G(v+t\phi)-G(\phi))\,dx - \int_{\mathbf{R}^N} V(x)g(v)\phi\,dx \\
&\quad = \int_0^1 ds \int_{\mathbf{R}^N} (g(v+ts\phi)-g(v))\phi\,dx.
\end{aligned} \tag{12}$$

We have

$$|(g(v+ts\phi)-g(v))\phi| \leqslant \begin{cases} C|\phi|^2, & |\phi|\leqslant 1, \\ C|\phi|, & |\phi|\geqslant 1, \end{cases} \tag{13}$$

so we have $|(g(v+ts\phi)-g(v))\phi| \leqslant CG(\phi)$, where we have used the fact: $|g(s)|\leqslant 2$, $|g'(s)| = |G''(s)| \leqslant 2$. By the dominated convergence theorem

$$\int_0^1 dx \int_{\mathbf{R}^N} (g(v+ts\phi)-g(v))\phi\,dx \to 0 \quad \text{as } t\to 0.$$

For the third term, we have

$$\frac{1}{t}\int_{\mathbf{R}^N}\frac{1}{p+1}(|f|^{p+1}(v+t\phi)-|f|^{p+1}(v))\,dx-\int_{\mathbf{R}^N}|f|^{p-1}ff'(v)\phi\,dx$$

$$=\int_0^1 ds\int_{\mathbf{R}^N}(|f|^{p-1}ff'(v+ts\phi)-|f|^{p-1}ff'(v))\phi\,dx$$

$$\leqslant\int_0^1 ds\left(\int_{\mathbf{R}^N}||f|^{p-1}ff'(v+ts\phi)-|f|^{p-1}ff'(v)|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}}||\phi||_{L^{2^*}}. \qquad (14)$$

Note that

$$||f|^{p-1}ff'(v+ts\phi)-|f|^{p-1}ff'(v)|$$

$$\leqslant C(|f|^{p-1}(v+\phi)+|f|^{p-1}(v)).$$

Hence, by the dominated convergence theorem

$$\lim_{t\to0}\frac{1}{t}\int_{\mathbf{R}^N}\frac{1}{p+1}(|f|^{p+1}(v+t\phi)-|f|^{p+1}(v))\,dx=\int_{\mathbf{R}^N}|f|^{p-1}ff'(v)\phi\,dx.$$

The Gateaux derivative $I'(v)$ has the form

$$\langle I'(v),\phi\rangle=\int_{\mathbf{R}^N}\nabla v\nabla\phi\,dx+\int_{\mathbf{R}^N}V(x)f(v)f'(v)\phi\,dx$$

$$-\int_{\mathbf{R}^N}|f|^{p-1}ff'(v)\phi\,dx.$$

By Propositions 2.1(5) and 2.3(1) and the fact that

$$\left|\int_{\mathbf{R}^N}|f|^{p-1}ff'(v)\phi\,dx\right|\leqslant C||f||^{p-1}_{L^{(p-1)\frac{2N}{N+2}}}||\phi||_{L^{2^*}},$$

we have that $I'(v)$ is a continuous linear functional on H^1_G.

Finally, the continuity with strong–weak topology is easy to check, as $v_n\to v$ in E_G, for any $\phi\in E_G$,

$$\int_{\mathbf{R}^N}(V(x)f(v_n)f'(v_n)-V(x)f(v)f'(v))\phi(x)\,dx\to0. \qquad \square$$

3. Existence results

We shall use the Mountain–Pass theorem (e.g., [1,28]) to prove the existence results. First we have the following about Palais-Smale sequences for I.

Proposition 3.1. (1) *Any* (PS) *sequence* $\{v_n\}$ *is bounded.*

(2) *If* $\{v_n\}$ *is a* (PS) *sequence and* $u_n = f(v_n)$ *converges in* L^{p+1}, *then* v_n *converges to* $v \in H^1_G$ in H^1_G. *Consequently*, $I(v) = \lim_{n\to\infty} I(v_n)$ *and* $I'(v) = 0$.

Proof. (1) Let $\{v_n\}$ be a (PS) sequence, $I(v_n) \to c \in \mathbf{R}$ and $I'(v_n) \to 0$ in the space $(H^1_G)^*$. We have

$$\begin{aligned} I(v_n) = &\frac{1}{2}\int_{\mathbf{R}^N} |\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbf{R}^N} V(x)G(v_n)\,dx \\ &- \frac{1}{p+1}\int_{\mathbf{R}^N} |f|^{p+1}(v_n)\,dx \\ = &\, c + o(1), \end{aligned} \tag{15}$$

$$\begin{aligned} \langle I'(v_n), \phi \rangle = &\int_{\mathbf{R}^N} \nabla v_n \nabla \phi\,dx + \int_{\mathbf{R}^N} V(x) f(v_n) f'(v_n)\phi\,dx \\ &- \int_{\mathbf{R}^N} |f|^{p-1} f f'(v_n)\phi\,dx \\ = &\, o(||\phi||). \end{aligned} \tag{16}$$

Choose $\phi = \frac{f(v_n)}{f'(v_n)} = \sqrt{1+f^2(v_n)}\,f(v_n)$. We have $|\phi| \leqslant C|v_n|$ and

$$|\nabla\phi| = \left(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\right)|\nabla v_n| \leqslant 2|\nabla v_n|$$

giving $||\phi|| \leqslant C||v_n||$. Thus, we have

$$\begin{aligned} &\int_{\mathbf{R}^N}\left(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\right)|\nabla v_n|^2\,dx + \int_{\mathbf{R}^N} V(x) f^2(v_n)\,dx - \int_{\mathbf{R}^N} |f|^{p+1}(v_n)\,dx \\ &\quad = o(||v_n||) \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\int_{\mathbf{R}^N}\left(\frac{1}{2} - \frac{1}{p+1}\left(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\right)\right)|\nabla v_n|^2\,dx \\ &\quad + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbf{R}^N} V(x) f^2(v_n)\,dx \\ &= c + o(1) + o(||v_n||). \end{aligned} \tag{18}$$

We consider first the case $4<p+1<\frac{4N}{N-2}$. For $p+1>4$ we have

$$\int_{\mathbf{R}^N} |\nabla v_n|^2\,dx + \int_{\mathbf{R}^N} V(x)f^2(v_n)\,dx \leqslant c + o(1) + o(||v_n||),$$

hence $\int_{\mathbf{R}^N} |\nabla v_n|^2 + \int_{\mathbf{R}^N} V(x)f^2(v_n)\,dx$ is bounded. Next if $p+1=4$, we have as above

$$\int_{\mathbf{R}^N}\left(1+\frac{f^2(v_n)}{1+f^2(v_n)}\right)|\nabla v_n|^2\,dx + \int_{\mathbf{R}^N} V(x)f^2(v_n)\,dx - \int_{\mathbf{R}^N} f^4(v_n)\,dx \\ = o(||v_n||) \tag{19}$$

and

$$\frac{1}{4}\int_{\mathbf{R}^N}\frac{|\nabla v_n|^2}{1+f^2(v_n)}\,dx + \frac{1}{4}\int_{\mathbf{R}^N} V(x)f^2(v_n)\,dx = c + o(1) + o(||v_n||). \tag{20}$$

If we denote $u_n = f(v_n)$, then $|\nabla v_n|^2 = (1+f^2(v_n))|\nabla u_n|^2$. Eqs. (19) and (20) can be written as

$$\int_{\mathbf{R}^N} (1+2u_n^2)|\nabla u_n|^2\,dx + \int_{\mathbf{R}^N} V(x)u_n^2 - \int_{\mathbf{R}^N} u_n^4\,dx \\ = o\left(\left(\int_{\mathbf{R}^N} (1+2u_n^2)|\nabla u_n|^2\,dx\right)^{\frac{1}{2}} + \left(\int_{\mathbf{R}^N} V(x)u_n^2\right)^{\frac{1}{2}}\right) \tag{21}$$

$$\frac{1}{8}\int_{\mathbf{R}^N} |\nabla u_n|^2\,dx + \frac{1}{8}\int_{\mathbf{R}^N} V(x)u_n^2\,dx \\ \leqslant c + o\left(\left(\int_{\mathbf{R}^N} u_n^2|\nabla u_n|^2\,dx\right)^{\frac{1}{2}}\right). \tag{22}$$

If $||v_n||$ is unbounded, the above two formulas imply for n large

$$\int_{\mathbf{R}^N} u_n^2|\nabla u_n|^2\,dx \leqslant c + \int_{\mathbf{R}^N} u_n^4\,dx. \tag{23}$$

By Sobolev embedding theorem and the Hölder inequality

$$\int_{\mathbf{R}^N} u_n^4\,dx \leqslant \left(\int_{\mathbf{R}^N} u_n^2\,dx\right)^{\frac{4}{N+2}} \left(\int_{\mathbf{R}^N} u_n^{\frac{4N}{N-2}}\,dx\right)^{\frac{N-2}{N+2}}$$

$$\leqslant C\left(\int_{\mathbf{R}^N} u_n^2\,dx\right)^{\frac{4}{N+2}} \left(\int_{\mathbf{R}^N} u_n^2|\nabla u_n|^2\,dx\right)^{\frac{N}{N+2}}$$

$$\leqslant C + o\left(\int_{\mathbf{R}^N} u_n^2|\nabla u_n|^2\,dx\right). \tag{24}$$

Here we have used formula (22). This together with (23) imply $\int_{\mathbf{R}^N} u_n^2|\nabla u_n|^2\,dx$ is bounded, hence so are $\int_{\mathbf{R}^N} |\nabla u_n|^2\,dx$, $\int_{\mathbf{R}^N} V(x)u_n^2\,dx$, and $||u_n||$.

To prove (2), up to a subsequence we have $v_n \to v$ a.e., $v_n \rightharpoonup v$ in $L^{\frac{2N}{N-2}}$ and $\nabla v_n \rightharpoonup \nabla v$ in L^2, and $f(v_n) \to f(v)$ in L^{p+1}. By the convexity of G we have

$$\left(\frac{1}{2}\int_{\mathbf{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbf{R}^N} V(x)G(v)\,dx\right) - \left(\frac{1}{2}\int_{\mathbf{R}^N} |\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbf{R}^N} V(x)G(v_n)\,dx\right)$$

$$\geqslant \int_{\mathbf{R}^N} (\nabla v - \nabla v_n)\nabla v_n\,dx + \int_{\mathbf{R}^N} V(x)f(v_n)f'(v_n)(v - v_n)\,dx$$

$$= -\int_{\mathbf{R}^N} |f|^{p-1}ff'(v_n)(v - v_n)\,dx + o(||v - v_n||) \to 0 \tag{25}$$

since $|f|^{p-1}ff'(v_n) \to |f|^{p-1}ff'(v)$ in $L^{\frac{2N}{N+2}}$ and $v_n \rightharpoonup v$ in $L^{\frac{2N}{N-2}}$. On the other hand, by the semicontinuity and Fatou's lemma,

$$\int_{\mathbf{R}^N} |\nabla v|^2\,dx \leqslant \liminf_{n\to\infty} \int_{\mathbf{R}^N} |\nabla v_n|^2\,dx$$

$$\int_{\mathbf{R}^N} V(x)G(v)\,dx \leqslant \liminf_{n\to\infty} \int_{\mathbf{R}^N} V(x)G(v_n)\,dx.$$

Therefore,

$$\int_{\mathbf{R}^N} |\nabla v|^2\,dx = \liminf_{n\to\infty} \int_{\mathbf{R}^N} |\nabla v_n|^2\,dx$$

$$\int_{\mathbf{R}^N} V(x)G(v)\,dx = \liminf_{n\to\infty} \int_{\mathbf{R}^N} V(x)G(v_n)\,dx.$$

By Proposition 2.1(3), $v_n \to v$ in E_G and we have $\nabla v_n \to \nabla v$ in L^2 too. Hence, $v_n \to v$ in H_G^1. $\square$

In order to produce a (PS) sequence we employ the Mountain–Pass theorem. Now let us define the Mountain–Pass value.

$$c_0 = \inf_{\gamma \in \Gamma} \sup_{i \in [0,\,1]} I(\gamma(t)), \tag{26}$$

where

$$\Gamma = \{\gamma \in C([0,1], H_G^1) \mid \gamma(0) = 0,\ I(\gamma(1)) \leqslant 0, \gamma(1) \neq 0\}. \tag{27}$$

To show c_0 is well defined we need to show that there exists $v \in H_G^1 \backslash \{0\}$ such that $I(v) \leqslant 0$. It suffices to find $u \in C_0^\infty(\mathbf{R}^N)$ such that $J(u) < 0$. Then $I(v) < 0$, where $v = h(u)$. When $p+1 > 4$, it is easy to see that for any $u \neq 0$, $J(tu) < 0$ for t large. When $p+1 = 4$, we take $u_0 \in C_0^\infty(\mathbf{R}^N)$ first and consider $u_s(x) = u_0(sx)$. Then it follows from a direct computation that for $s > 0$ small enough $(p+1)\int_{\mathbf{R}^N} u_s^2 |\nabla u_s|^2 < 2\int_{\mathbf{R}^N} |u_s|^4$. Fix such an $s > 0$ and we have $J(tu_s) < 0$ for t large.

Lemma 3.2. $c_0 > 0$.

Proof. Set

$$S_\rho = \left\{ v \in H_G^1 \mid \int_{\mathbf{R}^N} |\nabla v|^2\, dx + \int_{\mathbf{R}^N} V(x) f^2(v)\, dx = \rho^2 \right\}.$$

For $w = f(v)|f(v)|$ with $v \in S_\rho$, we have

$$\int_{\mathbf{R}^N} |\nabla w|^2\, dx = \int_{\mathbf{R}^N} \frac{4f^2(v)}{1+f^2(v)} |\nabla v|^2\, dx \leqslant 4\rho^2$$

$$\int_{\mathbf{R}^N} |w|\, dx \leqslant \int_{\mathbf{R}^N} V(x) f^2(v)\, dx \leqslant \rho^2,$$

$$\begin{aligned} \int_{\mathbf{R}^N} |f|^{p+1}(v)\, dx &= \int_{\mathbf{R}^N} |w|^{\frac{p+1}{2}}\, dx \\ &\leqslant \left(\int_{\mathbf{R}^N} |w|\, dx \right)^{\frac{\theta(p+1)}{2}} \left(\int_{\mathbf{R}^N} |w|^{2^*}\, dx \right)^{\frac{(1-\theta)(p+1)}{2}} \\ &\leqslant C\rho^{\frac{2\theta(p+1)}{2}} \left(\int_{\mathbf{R}^N} |\nabla w|^2 \right)^{\frac{2^*}{2}(1-\theta)\frac{p+1}{2}} \\ &\leqslant C\rho^{\frac{p+1}{2}(2\theta + 2^*(1-\theta))} = C\rho^{\frac{2N+2(p+1)}{N+2}}, \end{aligned} \tag{28}$$

where $\theta = \frac{2^* - \frac{p+1}{2}}{\frac{p+1}{2}(2^*-1)}$. Note that $\frac{2N+2(p+1)}{N+2} > 2$ if and only if $p+1>2$. Hence, for $v \in S_\rho$

$$I(v) \geqslant \frac{1}{2}\rho^2 - C\rho^{\frac{2N+2(p+1)}{N+2}} \geqslant \frac{1}{4}\rho^2$$

for $0<\rho<\rho_0 \ll 1$ for some ρ_0. If $\gamma(1)=v$ and $I(\gamma(1))<0$, then $\int_{\mathbf{R}^N} |\nabla v|^2 \, dx + \int_{\mathbf{R}^N} V(x) f^2(v) \, dx > \rho_0^2$ and

$$\sup_{t \in [0,1]} I(\gamma(t)) \geqslant \int \{I(w) \mid w \in S_{\rho_0}\} \geqslant \frac{1}{4}\rho_0^2 > 0.$$

Hence, $c_0 \geqslant \frac{1}{4}\rho_0^2 > 0$. $\square$

Proposition 3.3. *There is a $(PS)_{c_0}$ sequence $\{v_n\}$ for I with c being defined above.*

Proof. This follows with a standard argument from the Ekeland's variational principle and the strong–weak continuity of the Gauteaux derivative $I'(v)$.

Now we are ready to prove our main result Theorem 1.1. The compact case is considered first where we assume (V1) or (V2).

Proof of Theorem 1.1. Cases (V1) or (V2): By Proposition 3.3 we have a $(\mathrm{PS})_{c_0}$ sequence $\{v_n\}$ for the Mountain–Pass value $c_0>0$. By Proposition 3.1(1) $\{v_n\}$ is bounded in H_G^1. By Proposition 3.1(2) v_n converges to v in H_G^1 if $f(v_n) \to f(v)$ in L^{p+1}. By Proposition 2.2, under (V1) or (V2) with $N \geqslant 2$, the map $v \to f(v)$ from H_G^1 into L^{p+1} is compact. Up to a subsequence, we have $v_n \to v$ in H_G^1 and v is a critical point of I. Since we may replace v_n by $|v_n|$ we may assume $v \geqslant 0$ in $\mathbf{R}^N$. By elliptic regularity theory, we have v smooth. By Lemma 3.7 below $v>0$ in $\mathbf{R}^N$. $\square$

Cases (V3) and (V4) are locally compact ones for which we need more arguments. First by using the concentration–compactness lemma due to P. L. Lions we have the following lemma.

Lemma 3.4. *Let $v_n \subset H_G^1$ be a $(PS)_{c_0}$ sequence for I. Then by passing to a subsequence we have $A := \lim_{n\to\infty} \int_{\mathbf{R}^N} |f(v_n)|^{p+1} \, dx > 0$. Furthermore, there is $\beta \in (0,1]$ and $x_n \subset \mathbf{R}^N$ such that for any $\varepsilon>0$ there exists $R>0$, for any $R' \geqslant R$ it holds*

$$\liminf_{n\to\infty} \int_{B_R(x_n)} |f(v_n)|^{p+1} \, dx \geqslant \beta A - \varepsilon,$$

$$\liminf_{n\to\infty} \int_{\mathbf{R}^N \setminus B_{R'}(x_n)} |f(v_n)|^{p+1} \, dx \geqslant (1-\beta)A - \varepsilon.$$

Proof. First by Proposition 3.1), (v_n) is bounded and therefore $\int_{\mathbf{R}^N} |f(v_n)|^{p+1}\,dx$ is bounded. We claim that $\int_{\mathbf{R}^N} |f(v_n)|^{p+1}\,dx$ is bounded away from zero. Otherwise, $f(v_n)\to 0$ in L^{p+1}, then $v_n \to 0$ in H^1_G by Proposition 2.1(2). Hence, $I(v_n)\to 0$, which contradicts with $c_0>0$. Let $w_n = f(v_n)^2$. We apply the concentration–compactness lemma due to Lions [33] to $|w_n|^{\frac{p+1}{2}} = |f(v_n)|^{p+1}$. By Proposition 2.2, w_n is bounded in L^{2^*} and thus bounded in L^2. Since a direct computation shows that $\int_{\mathbf{R}^N} |\nabla w_n|^2 \leqslant \int_{\mathbf{R}^N} |\nabla v_n|^2$, we get that w_n is bounded in $H^1(\mathbf{R}^N)$. Then by a Lemma due to Lions (e.g., [34]) the vanishing case cannot occur. Then it follows from the concentration–compactness lemma that there is $\beta>0$, x_n such that the conclusion holds. $\square$

Lemma 3.5. $\beta = 1$ *in the previous lemma*, *i.e.*, *the dichotomy cannot occur*.

Before proving Lemma 3.5, we give a remark.

Remark 3.6. If $\beta = 1$, by Lemma 3.5 and the compactness of the map: $v \to f(v)$ from $H^1_G \to L^{p+1}_{\mathrm{loc}}(\mathbf{R}^N)$ we have $f(v_n)\to f(v)$ in L^{p+1}, hence by Proposition 3.1 $v_n \to v$.

Proof of Lemma 3.5. Let η and ζ be smooth functions, satisfying $\eta(s) = 1$ for $s\leqslant 2R$, $\eta(s) = 0$ for $s\geqslant 3R$; $\zeta(s) = 1$ for $2R\leqslant s\leqslant 3R$, $\zeta(s) = 0$ for $s\leqslant R$ or $s\geqslant 4R$. We claim

$$\int_{T_R} |\nabla v_n|^2\,dx + \int_{T_R} V(x) f^2(v_n)\,dx + \int_{T_R} |f(v_n)|^{p+1}\,dx = o_R(1) + o_n(1) + O(\varepsilon),$$

where $T_R = T_R(x_n) = \{x \mid 2R\leqslant |x - x_n| \leqslant 3R\}$, $o_R(1)\to 0$ as $R\to\infty$ uniformly in n, $o_n(1)\to 0$ as $n\to\infty$. To see these, let $\phi = \zeta^2(|x-x_n|) f(v_n)\sqrt{1+f^2(v_n)}$. Using $\nabla\zeta = O(\frac{1}{R})$ we have $||\phi||$ is bounded and $\langle I'(v_n), \phi\rangle = o_n(1)$. A direct computation shows

$$\int_{\mathbf{R}^N} \zeta^2 |\nabla v_n|^2\,dx + \int_{\mathbf{R}^N} V(x)\zeta^2 f^2(v_n)\,dx = \int_{\mathbf{R}^N} \zeta^2 |f(v_n)|^{p+1}\,dx + O\left(\frac{1}{R}\right) + o_n(1).$$

Here we have used the fact that $\int_{B_{4R}\setminus B_R(x_n)} |f(v_n)|^{p+1}\,dx \leqslant 2\varepsilon$.

Next we define $w_n = \eta(|x-x_n|)v_n$, $z_n = (1-\eta(|x-x_n|))v_n$. Then

$$\int_{\mathbf{R}^N} f^{p+1}(w_n)\,dx \geqslant \int_{B_{2R}(x_n)} f^{p+1}(v_n)\,dx \geqslant \beta A - \varepsilon,$$

$$\int_{\mathbf{R}^N} f^{p+1}(z_n)\,dx \geqslant \int_{\mathbf{R}^N \setminus B_{3R}(x_n)} f^{p+1}(v_n)\,dx \geqslant (1-\beta)A - \varepsilon.$$

By the claim above, we have

$$I(v_n) = I(w_n) + I(z_n) + o_n(1) + O_R(1) + O(\varepsilon).$$

We claim that $I(w_n) \geqslant c_0 + o(1)$ and $I(z_n) \geqslant c_0 + o(1)$. Taking limit, we get a contradiction $c_0 \geqslant 2c_0$. To see this claim, we construct a curve $\gamma : [0,1] \to H^1_G$ with $\gamma(0) = 0$, $\gamma(1) < 0$ and $\sup_{t\in[0,1]} I(\gamma(t)) = I(w_n) + o(1)$ (or $\sup_{t\in[0,1]} I(\gamma(t)) = I(z_n) + o(1)$, respectively). First we have the following estimate:

$$\begin{aligned} o(1) &= \langle I'(w_n), \sqrt{1+f^2(w_n)}f(w_n)\rangle \\ &= \int_{\mathbf{R}^N} \left(1 + \frac{f^2(w_n)}{1+f^2(w_n)}\right)|\nabla w_n|^2\,dx + \int_{\mathbf{R}^N} V(x)f^2(w_n)\,dx \\ &\quad - \int_{\mathbf{R}^N} |f(w_n)|^{p+1}\,dx. \end{aligned} \tag{29}$$

This follows from

$$\begin{aligned} o(1) &= \langle I'(v_n), \eta^2\sqrt{1+f^2(v_n)}f(v_n)\rangle \\ &= \int_{\mathbf{R}^N} \eta^2\left(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\right)|\nabla v_n|^2\,dx + \int_{\mathbf{R}^N} V(x)f^2(v_n)\eta^2\,dx \\ &\quad - \int_{\mathbf{R}^N} \eta^2|f(v_n)|^{p+1}\,dx \end{aligned} \tag{30}$$

and the fact that $w_n = v_n$ for $|x - x_n| \leqslant 2R$ and $w_n = 0$ for $|x - x_n| \geqslant 3R$. Now let $u_n = f(w_n)$ the above implies

$$\int_{\mathbf{R}^N} (1 + 2u^2)|\nabla u_n|^2\,dx + \int_{\mathbf{R}^N} V(x)u_n^2\,dx - \int_{\mathbf{R}^N} |u_n|^{p+1}\,dx = o(1).$$

Next, we claim that there exist $t_n \to 1$ such that

$$\begin{aligned} &\int_{\mathbf{R}^N} (1 + 2(t_nu_n)^2)|\nabla t_nu_n|^2\,dx + \int_{\mathbf{R}^N} V(x)(t_nu_n)^2\,dx \\ &\quad - \int_{\mathbf{R}^N} |t_nu_n|^{p+1}\,dx = 0. \end{aligned}$$

To see this, let us denote

$$\begin{aligned} a_n &= \int_{\mathbf{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbf{R}^N} V(x)u_n^2\,dx, \quad b_n = 2\int_{\mathbf{R}^N} u_n^2|\nabla u_n|^2\,dx, \\ c_n &= \int_{\mathbf{R}^N} |u_n|^{p+1}\,dx. \end{aligned}$$

Then $a_n + b_n - c_n = o(1)$, $c_n \to c_* \in (0, \infty)$, $a_n \to a_* \in [0, \infty)$, and $b_n \to b_* \in [0, \infty)$. If $a_* = 0$, by using that b_n is bounded, we have $c_n \to 0$, so $a_* > 0$. Then for n large we can solve equation $a_n + t^2 b_n - t^{p-1} c_n = 0$ which has a unique positive solution t_n converging to the unique positive solution t_* for equation $a_* + t^2 b_* - t^{p-1} c_* = 0$. Since we already know $a_* + b_* - c_* = 0$, in case $p+1 > 4$, we must have $t_* = 1$. When $p+1 = 4$, we need to solve $a_n + t^2 b_n - t^2 c_n = 0$. Since $a_n \to a_* > 0$, $a_* + b_* - c_* = 0$, we have $c_n > b_n$ for n large. So again we get a unique solution t_n for the equation $a_n + t^2 b_n - t^2 c_n = 0$.

Finally, let $\gamma(t) = h(tu_n)$. We claim

$$\sup_{t \geqslant 0} I(\gamma(t)) = I(\gamma(t_n)) = J(t_n u_n).$$

To see this,

$$\begin{aligned} I(\gamma(t)) &= J(tu_n) \\ &= \frac{1}{2} \int_{\mathbf{R}^N} (1 + t^2 u_n^2) t^2 |\nabla u_n|^2 \, dx + \frac{1}{2} \int_{\mathbf{R}^N} V(x) t^2 u_n^2 \, dx \\ &\quad - \frac{1}{p+1} \int_{\mathbf{R}^N} t^{p+1} |u_n|^{p+1} \, dx. \end{aligned} \tag{31}$$

Using $\frac{d}{dt} J(tu_n) = 0$ we have

$$\int_{\mathbf{R}^N} (1 + 2t^2 u_n^2) |\nabla u_n|^2 \, dx + \int_{\mathbf{R}^N} V(x) u_n^2 \, dx - \int_{\mathbf{R}^N} t^{p-2} |u_n|^{p+1} \, dx.$$

Hence, $t = t_n$. By a rescaling, we find a curve $\gamma(t)$ such that $\sup_{t \in [0,1]} I(\gamma(t)) = J(t_n u_n) = J(u_n) + o(1)$, hence $I(w_n) \geqslant c_0 + o(1)$. Similarly, if $\beta < 1$ we have $I(z_n) \geqslant c_0 + o(1)$. This proves that $\beta = 1$. $\square$

Proof of Theorem 1.1. Case (V3): By Proposition 3.1, there is a $(\mathrm{PS})_{c_0}$ sequence $\{v_n\} \subset H_G^1$ for I, i.e., $I(v_n) \to c_0$ and $I'(v_n) \to 0$ as $n \to \infty$. By Lemma 3.5 there exist $\{y_n\} \subset \mathbf{R}^N$ such that for any $\varepsilon > 0$ there is $R > 0$

$$\liminf \int_{B_R(y_n)} |f(v_n)|^{p+1} \, dx \geqslant \lim \int_{\mathbf{R}^N} |f(v_n)|^{p+1} \, dx - \varepsilon.$$

Since V is periodic in each variable $x_1, \ldots, x_n$, we may assume that $\{y_n\}$ is bounded. By the locally compact map $v \to f(v)$ from H_G^1 into L_{loc}^{p+1}, we have the convergence $f(v_n) \to f(v)$ in $L^{p+1}(\mathbf{R}^N)$, hence $v_n \to v$ in H_G^1, and $I'(v) = 0$, $I(v) = c_0$. Similar to the case (V1), we have $v > 0$ in $\mathbf{R}^N$. The proof is complete. $\square$

Proof of Theorem 1.1. Case (V4): Consider the limit functional

$$I_\infty(v) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla v|^2 + \int_{\mathbf{R}^N} V_\infty f^2(v) - \int_{\mathbf{R}^N} |f(v)|^{p+1}$$

where $V_\infty = \lim_{|x|\to\infty} V(x) = \sup_{\mathbf{R}^N} V(x)$. Define

$$c_\infty = \inf_{\gamma\in\Gamma_\infty} \sup_{t\in[0,1]} I_\infty(\gamma(t))$$

where $\Gamma_\infty = \{\gamma\in C([0,1], H_G^1 \mid \gamma(0) = 0, I_\infty(\gamma(1))<0\}$. By Theorem 1.1 under (V3), I_∞ has a critical point $v_\infty \in H_G^1$, $I_\infty(v_\infty) = c_\infty$, $I'_\infty(v_\infty) = 0$. Let $u_\infty = f(v_\infty)$, and $\gamma(t) : [0,1] \to H_G^1$, $\gamma(t) = h(tu_\infty)$. Then $\sup_{t\in[0,1]} I(\gamma(t)) = I_\infty(\gamma(1)) = c_\infty$. It is clear that $I(\gamma(t)) \leqslant I_\infty(\gamma(t))$. The strict inequality holds if either $V(x) < V_\infty$ for all $x\in\mathbf{R}^N$; or $V(x) \equiv V_\infty$ and $v_\infty(x) > 0$ in $\mathbf{R}^N$. We assume the stronger condition $V(x) < V_\infty$. We have $I(\gamma(t)) \leqslant I_\infty(\gamma(t))$, hence $c \leqslant \sup_t I(\gamma(t)) < c_\infty = \sup_t I_\infty(\gamma(t))$. As in the case (V3), we have a $(PS)_{c_0}$ sequence $\{v_n\}\in H_G^1$ such that $I(v_n)\to c_0$, $I'(v_n)\to 0$, and a sequence $\{y_n\}\in\mathbf{R}^N$ such that for any $\varepsilon>0$ there is $R>0$ such that

$$\liminf \int_{B_R(y_n)} |f(v_n)|^{p+1}\,dx \geqslant \lim \int_{\mathbf{R}^N} |f(v_n)|^{p+1}\,dx - \varepsilon.$$

As in case (V3), we need to prove the boundedness of the sequence $\{y_n\}$ only. We have

$$\begin{aligned}&\frac{1}{2}\int_{\mathbf{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbf{R}^N} V(x) f^2(v_n)\,dx - \frac{1}{p+1}\int_{\mathbf{R}^N} |f(v_n)|^{p+1}\\ &\quad = c_0 + o(1),\end{aligned}$$

$$\begin{aligned}&\int_{\mathbf{R}^N}\left(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\right)|\nabla v_n|^2\,dx + \int_{\mathbf{R}^N} V(x) f^2(v_n)\,dx\\ &\quad - \int_{\mathbf{R}^N} |f(v_n)|^{p+1}\,dx = o(1).\end{aligned}$$

Suppose that $|y_n|\to\infty$. Then

$$\begin{aligned}&\int_{\mathbf{R}^N} V_\infty f^2(v_n)\,dx - \int_{\mathbf{R}^N} V(x) f^2(v_n)\,dx\\ &\quad = \int_{\mathbf{R}^N\setminus B_R(y_n)} (V_\infty - V(x)) f^2(v_n)\,dx + \int_{B_R(y_n)} (V_\infty - V(x)) f^2(v_n)\,dx\\ &\quad \leqslant C\int_{\mathbf{R}^N\setminus B_R(y_n)} f^2(v_n)\,dx + C \sup_{|x|\geqslant |y_n|-R} |V_\infty - V(x)|\\ &\quad \to 0. \qquad (32)\end{aligned}$$

Hence

$$\begin{aligned} I_\infty(v_n) &= \frac{1}{2}\int_{\mathbf{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbf{R}^N} V_\infty f^2(v_n)\,dx - \frac{1}{p+1}\int_{\mathbf{R}^N} |f(v_n)|^{p+1} \\ &= c_0 + o(1). \end{aligned} \tag{33}$$

$$\begin{aligned} &\int_{\mathbf{R}^N}\left(1+\frac{f^2(v_n)}{1+f^2(v_n)}\right)|\nabla v_n|^2\,dx + \int_{\mathbf{R}^N} V_\infty f^2(v_n)\,dx \\ &\quad - \int_{\mathbf{R}^N} |f(v_n)|^{p+1}\,dx = o(1). \end{aligned}$$

These two formulas imply that $c_\infty \leqslant c$, as we have done for the functional I. This contradicts our previous conclusion $c_\infty > c_0$. Again, we can argument that $v>0$ in $\mathbf{R}^N$ as before. The proof is complete. $\square$

Finally, we have the following lemma which gives the positivity of the solutions we obtained.

Lemma 3.7. *A nonnegative solution v of*

$$-\Delta v + V(x)f(v)f'(v) = |f(v)|^{p-1}f(v)f'(v)$$

is positive in $\mathbf{R}^N$.

Proof. Note that near $v=0$, we have $f(v)\sim v$, $f'(v)\sim 1$. The equation is of form

$$-\Delta v + k(x)v = 0,$$

where $k(x)>0$ in a neighborhood $B_r(x)$ of a point x with $v(x)=0$. Now applying Hopf's lemma we know v has to be strictly positive. $\square$

Remark 3.8. A minimization approach was used in [22,25] for equation (2), and solutions were obtained for a sequence of λ_n as Lagrange multipliers [10] tending to 0 and ∞. Here we obtain solutions for all $\lambda>0$.

Remark 3.9. In [22] a more general equation was considered

$$-\Delta u + V(x)u - (\Delta(|u|^{2\alpha}))|u|^{2\alpha-2}u = \lambda|u|^{p-1}u, \quad u>0 \text{ in } \mathbf{R}^N, \tag{34}$$

where $\alpha>\frac{1}{2}$. Our method in this paper can be adopted to this more general problem to get the following result.

Theorem 3.10. *Let $\alpha>\frac{1}{2}$ and $4\alpha\leqslant p+1<2\alpha 2^*$. Then for any $\lambda>0$,* (34) *has a positive solution, provided that one of the following four conditions hold*: (V1); (V2) *and* $N\geqslant 2$; (V3); (V4).

Remark 3.11. Our method applies to more general nonlinearity which are not necessarily of pure power form. It does not need major changes of our arguments. We leave the statements of the results to interested readers. (V1) can be replaced by any conditions for compactness from X into L^q for $2 \leqslant q < 2^*$ (see [4,16]).

Remark 3.12. In recent years, critical point theory for continuous functionals have been developed (e.g., [2,3,11,13] and references therein) which could be used directly for our problem by using an approximation argument. However, it seems that using this type of theory one has to further restrict the growth of the nonlinearity, namely, $p+1 < 2^*$, which together with $p+1 \geqslant 4$ would limit the space dimensions to $N = 1, 2, 3$ only.

Remark 3.13. It is interesting to note that $22^* = \frac{4N}{N-2}$ behaves like a critical exponent for Eq. (2). We do not know whether there are solutions for $p+1 = 22^*$.

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